A SHORT NOTE ON CONVEX FUNCTIONS

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Abstract. This short note concerns three well-known inequalities for convex functions and their interesting proofs and applications I discovered back when I was an undergrad.

In [1] Alzera gave a simple and elegant proof of the classical arithmetic mean-geometric mean inequality [2, Theorem 9]:

$$\prod_{i=1}^{n} a_i^{p_i} \le \sum_{i=1}^{n} p_i a_i,$$

where $a_1, ..., a_n$ and $p_1, ..., p_n$ are positive real numbers with $\sum_{i=1}^{n} p_i = 1$. We now show that his method can be used to prove Jensen's inequality for convex functions [2, Theorem 90]. Recall a function $f\colon [a,b] \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in [a,b]$ and any $\lambda \in [0,1]$. Moreover, f is called strictly convex if it is convex and

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$$

implies $x = y$ or $\lambda \in \{0,1\}$. Now we prove the following result.

Proposition 1. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable convex function. Then*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

for all real numbers $x_1, ..., x_n \in [a,b]$ and positive real numbers $p_1, ..., p_n$ with $\sum_{i=1}^{n} p_i = 1$. Moreover, if f is strictly convex on $[a,b]$, then equality in (1) holds if and only if $x_1 = ... = x_n$.

Proof. Without loss of generality, assume $x_1 \le ... \le x_n$. Let $A := \sum_{i=1}^{n} p_i x_i$. Since f is differentiable and convex, it is continuously differentiable. Suppose $|f'(x)| < M$ for all $x \in [a,b]$. Let $g(x) := f(x) + Mx$ for $x \in [a,b]$. Then g is convex on $[a,b]$ and $g'(x) = f'(x) + M > 0$ for all $x \in [a,b]$. This implies that g is strictly increasing on $[a,b]$. Observe

$$g(x_1) \le \sum_{i=1}^{n} p_i g(x_i) \le g(x_n).$$

Thus there exists a unique $B \in [x_1, x_n]$ for which

$$g(B) = \sum_{i=1}^{n} p_i g(x_i).$$

Furthermore, there exists $1 \leq k < n$ for which $x_k \leq B \leq x_{k+1}$. Since g is convex on $[a, b]$, we have that g' is increasing on $[a, b]$. It follows that

$$\sum_{i=1}^{k} p_i \int_{x_i}^{B} (g'(B) - g'(x))\, dx + \sum_{i=k+1}^{n} p_i \int_{B}^{x_i} (g'(x) - g'(B))\, dx \geq 0. \tag{2}$$

The left-hand side is

$$\sum_{i=1}^{n} p_i \int_{x_i}^{B} g'(B)\, dx - \sum_{i=1}^{n} p_i \int_{x_i}^{B} g'(x)\, dx = (B - A)g'(B).$$

Since $g'(B) > 0$, we have $A \leq B$. This is equivalent to $g(A) \leq g(B)$. But note that $g(A) = f(A) + MA$ and

$$g(B) = \sum_{i=1}^{n} p_i g(x_i) = \sum_{i=1}^{n} p_i f(x_i) + MA.$$

This gives (1).

Suppose further that f is strictly convex on $[a, b]$. Then g is also strictly convex on $[a, b]$. Thus g' is strictly increasing on $[a, b]$. If equality in (1) holds, then $A = B$. This implies that equality in (2) holds, and hence we must have $x_i = B$ for all $1 \leq i \leq n$. Thus $x_1 = ... = x_n$. This completes the proof. □

By a similar but slightly complicated argument we can prove the following result which generalizes an inequality of Szegö [4] when the convex function considered is differentiable.

Proposition 2. *Let $f\colon [0, a] \to \mathbb{R}$ be a differentiable convex function. Let $n \geq 1$ be an odd integer and let $x_1, ..., x_n \in [0, a]$ be real numbers with $x_1 \geq ... \geq x_n$. Suppose that $p_1, ..., p_n \in [0, 1]$ are non-negative real numbers such that $p_1 \geq ... \geq p_n$. Then*

$$f\left(\sum_{i=1}^{n} (-1)^{i-1} p_i x_i\right) \leq \sum_{i=1}^{n} (-1)^{i-1} p_i f(x_i) + \lambda f(0), \tag{3}$$

where

$$\lambda = 1 - \sum_{i=1}^{n} (-1)^{i-1} p_i.$$

Proof. Let $y_i := x_i$ for all $1 \leq i \leq n$ and $y_{n+1} := 0$. Put $p_{n+1} := -\lambda$. Then we have

$$\sum_{i=1}^{n+1} (-1)^{i-1} p_i = 1 \tag{4}$$

and

$$\sum_{i=1}^{n+1} (-1)^{i-1} p_i y_i = \sum_{i=1}^{n} (-1)^{i-1} p_i x_i. \tag{5}$$

Without loss of generality, we may suppose that $f'(x) > 0$ for all $x \in [0, a]$. Let $A := \sum_{i=1}^{n} (-1)^{i-1} p_i x_i$. Since $x_1 \geq ... \geq x_n \geq 0$ and $1 \geq p_1 \geq ... \geq p_n \geq 0$, we see that

$$0 \leq p_n x_n \leq A \leq p_1 x_1 \leq x_1.$$

Since f is increasing on $[0, a]$, we have

$$\sum_{i=1}^{n}(-1)^{i-1}p_i(f(x_i) - f(0)) \le p_1(f(x_1) - f(0)) \le f(x_1) - f(0).$$

It follows that

$$\sum_{i=1}^{n}(-1)^{i-1}p_i f(x_i) + \lambda f(0) = \sum_{i=1}^{n}(-1)^{i-1}p_i(f(x_i) - f(0)) + f(0) \le f(x_1).$$

By partial summation we have

$$\sum_{i=1}^{n}(-1)^{i-1}p_i f(x_i) = \sum_{i=1}^{n-1}\sum_{j=1}^{i}(-1)^{j-1}p_j(f(x_i) - f(x_{i+1})) + (1-\lambda)f(x_n) \ge (1-\lambda)f(x_n).$$

Hence we have

$$\sum_{i=1}^{n}(-1)^{i-1}p_i f(x_i) + \lambda f(0) \ge (1-\lambda)f(x_n) + \lambda f(0) \ge f(0).$$

Thus there exists a unique $B \in [y_{n+1}, y_1]$ for which

$$f(B) = \sum_{i=1}^{n}(-1)^{i-1}p_i f(x_i) + \lambda f(0) = \sum_{i=1}^{n+1}(-1)^{i-1}p_i f(y_i). \tag{6}$$

Moreover, there exists $1 \le k \le n$ for which $y_{k+1} \le B \le y_k$. Since f is convex on $[0, a]$, we have that f' is increasing on $[0, a]$. Note that

$$\int_B^{y_i} (f'(x) - f'(B))\, dx$$

is non-negative and decreases as i increases from 1 to k. Hence we have

$$\sum_{i=1}^{k}(-1)^{i-1}p_i \int_B^{y_i} (f'(x) - f'(B))\, dx \ge 0. \tag{7}$$

Similarly, we see that

$$h_i := \int_{y_{n+1-i}}^{B} (f'(B) - f'(x))\, dx$$

is non-negative and decreases as i increases from 0 to $n-k$. Note that

$$\sum_{i=k+1}^{n+1} (-1)^{i-1}p_i \int_{y_i}^{B} (f'(B) - f'(x))\, dx = \sum_{i=0}^{n-k}(-1)^{n-i}p_{n+1-i}h_i.$$

By partial summation and (4) we see that the right-hand side is

$$\begin{aligned}
&\sum_{i=0}^{n-k-1}\left(\sum_{j=0}^{i}(-1)^{n-j}p_{n+1-j}\right)(h_i-h_{i+1})+\left(\sum_{i=0}^{n-k}(-1)^{n-i}p_{n+1-i}\right)h_{n-k}\\
&=\sum_{i=0}^{n-k-1}\left(1-\sum_{j=1}^{n-i}(-1)^{j-1}p_j\right)(h_i-h_{i+1})+\left(1-\sum_{i=1}^{k}(-1)^{i-1}p_i\right)h_{n-k}\\
&=h_0-\sum_{i=0}^{n-k-1}\left(\sum_{j=1}^{n-i}(-1)^{j-1}p_j\right)(h_i-h_{i+1})-\sum_{i=1}^{k}(-1)^{i-1}p_ih_{n-k}\\
&\geq h_0-\sum_{i=0}^{n-k-1}p_1(h_i-h_{i+1})-p_1h_{n-k}\\
&=0.
\end{aligned}$$

Hence we have

$$\sum_{i=k+1}^{n+1}(-1)^{i-1}p_i\int_{y_i}^{B}(f'(B)-f'(x))\,dx\geq 0. \tag{8}$$

Adding up (7) and (8) we obtain

$$\sum_{i=1}^{k}(-1)^{i-1}p_i\int_{B}^{y_i}(f'(x)-f'(B))\,dx+\sum_{i=k+1}^{n+1}(-1)^{i-1}p_i\int_{y_i}^{B}(f'(B)-f'(x))\,dx\geq 0.$$

In view of (4)–(6), the left-hand side equals $(B-A)f'(B)$. Since $f'(B)>0$, we have $A\leq B$. Since f is increasing on $[0,a]$, we conclude that $f(A)\leq f(B)$. This proves (3). □

Remark 1. Szegö's inequality follows from Proposition 2 by taking $p_1=...=p_n=1$, at least when the convex function in consideration is differentiable.

Remark 2. If $f\colon[a,b]\to\mathbb{R}$ is convex but not necessarily differentiable, then f is continuous on the open interval (a,b) and admits left and right derivatives both of which are increasing on (a,b). Moreover, f is differentiable everywhere on (a,b) except for a subset $E\subseteq(a,b)$ that is at most countable. Hence the arguments used in the proofs of Propositions 1 and 2 may be adapted to accommodate this general case.

We obtain the following result as a corollary of Proposition 2.

Corollary 3. *Let p,q be real numbers with $|p|\geq|q|>0$. Let $n\geq 1$ be an odd integer and let $x_1,...,x_n$ be positive real numbers with $x_1\geq...\geq x_n$. Then*

$$\operatorname{sgn}(p)\left(\sum_{i=1}^{n}(-1)^{i-1}x_i^p\right)^{1/p}\geq\operatorname{sgn}(p)\left(\sum_{i=1}^{n}(-1)^{i-1}x_i^q\right)^{1/q},$$

where

$$\operatorname{sgn}(p):=\begin{cases}1 & \text{if } p>0,\\ -1 & \text{if } p<0.\end{cases}$$

Proof. Let $r := p/q$ and $f(x) := (x+\varepsilon)^r$ for $x \geq 0$, where $\varepsilon > 0$. Then

$$f''(x) = r(r-1)(x+\varepsilon)^{r-2} \geq |r|(|r|-1)(x+\varepsilon)^{r-2} \geq 0$$

for all $x \geq 0$. Thus f is convex on $(0, +\infty)$. Our corollary follows by applying Proposition 2 to f and $x_1^q, ..., x_n^q$ and letting $\varepsilon \to 0$. $\square$

Both Jensen's inequality and Szegö's inequality follow from a beautiful result of Karamata [3]. Let $f\colon [a,b] \to \mathbb{R}$ be a convex function and let $p_1, ..., p_n$ be positive real numbers. Let $x_1, ..., x_n \in [a,b]$ and $y_1, ..., y_n \in [a,b]$ be two sequences of real numbers arranged in descending order, that is, $x_1 \geq ... \geq x_n$ and $y_1 \geq ... \geq y_n$, such that

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i \tag{9}$$

and

$$\sum_{i=1}^{k} p_i x_i \geq \sum_{i=1}^{k} p_i y_i \tag{10}$$

for all $1 \leq k < n$. Karamata's inequality in its weighted form states that under these conditions, we have

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{i=1}^{n} p_i f(y_i).$$

The classical proofs use partial summation. We give a simple proof of Karamata's inequality for differentiable convex functions. We first prove the following variant of Karamata's inequality.

Proposition 4. *Let $f_1(x), ..., f_n(x)$ be differentiable convex functions on $[a,b]$ such that $f_1'(x) \geq ... \geq f_n'(x) \geq 0$ for all $x \in [a,b]$. Let $p_1, ..., p_n$ be positive real numbers. Suppose that $x_1, ..., x_n \in [a,b]$ and $y_1, ..., y_n \in [a,b]$ are two sequences of real numbers arranged in descending order satisfying (10) for all $1 \leq k \leq n$. Then*

$$\sum_{i=1}^{n} p_i f_i(x_i) \geq \sum_{i=1}^{n} p_i f_i(y_i). \tag{11}$$

Furthermore, if $f_1, ..., f_n$ are strictly convex on $[a,b]$ with $f_n'(x) > 0$ for all $x \in [a,b]$, then equality in (11) holds if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. For each $1 \leq k \leq n$, let

$$A_k := \sum_{i=1}^{k} p_i (x_i - y_i),$$

$$B_k := \sum_{i=1}^{k} p_i (f_i(x_i) - f_i(y_i)).$$

Since $A_n \geq 0$ and $f_n'(x) \geq 0$ for all $x \in [a,b]$, it is sufficient to show

$$B_n \geq f_n'(y_n) A_n. \tag{12}$$

For any convex function $f\colon [a,b] \to \mathbb{R}$, we have that f' is increasing on $[a,b]$ and

$$f(x) - f(y) \geq f'(y)(x-y) \tag{13}$$

for all $x, y \in [a, b]$. Thus $B_1 \geq f_1'(y_1)A_1$. Since $A_1 \geq 0$ and $f_1'(y_1) \geq f_1'(y_2) \geq f_2'(y_2)$, we have

$$B_2 \geq B_1 + f_2'(y_2)p_2(x_2 - y_2) \geq f_2'(y_2)(A_1 + p_2(x_2 - y_2)) = f_2'(y_2)A_2.$$

The inequality (12) follows by iterating this procedure. The assertion for equality in (11) follows from the observation that if f is strictly convex on $[a, b]$ with $f'(x) > 0$ for all $x \in [a, b]$, then equality in (13) occurs precisely when $x = y$. □

We obtain the following result as a corollary of Proposition 4 of which Karamata's inequality for differentiable convex functions is a special case.

Proposition 5. *Let $f_1(x), ..., f_n(x)$ be differentiable convex functions on $[a, b]$ such that $f_1'(x) \geq ... \geq f_n'(x)$ for all $x \in [a, b]$. Let $p_1, ..., p_n$ be positive real numbers. Suppose that $x_1, ..., x_n \in [a, b]$ and $y_1, ..., y_n \in [a, b]$ are two sequences of real numbers arranged in descending order satisfying (9) and (10). Then*

$$\sum_{i=1}^{n} p_i f_i(x_i) \geq \sum_{i=1}^{n} p_i f_i(y_i). \tag{14}$$

Furthermore, if $f_1, ..., f_n$ are strictly convex on $[a, b]$, then equality in (14) holds if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. This follows from Proposition 4 applied to $g_i(x) := f_i(x) + Mx$ for $1 \leq i \leq n$, where $M > 0$ is any positive real number for which $|f_n'(x)| < M$ holds for all $x \in [a, b]$. □

We now give a few simple applications of the inequalities of Jensen and Karamata which may be of some interest.

Corollary 6. *Let $z_1, ..., z_n \in \mathbb{C}$ be complex numbers in the open unit disk $D := \{z \in \mathbb{C} \colon |z| < 1\}$. Suppose that $p_1, ..., p_n$ are non-negative real numbers with $\sum_{j=1}^{n} p_j = 1$. Then*

$$\prod_{j=1}^{n} (1 + z_j)^{p_j} - 1 \in D.$$

Here $(1 + z_j)^{p_j} = \exp(p_j \log(1 + z_j))$ with log *taken to be the same branch of the natural logarithm for all $1 \leq j \leq n$.*

Proof. Since $\sum_{j=1}^{n} p_j = 1$, we may suppose that $\log z$ takes values of the principle branch. Let $1 + z_j = r_j e^{i\theta_j}$, where $\theta_j \in (-\pi, \pi]$. Since $|z_j| < 1$, we deduce that $\theta_j \in (-\pi/2, \pi/2)$ and $r_j < 2\cos\theta_j$. The function $f(x) := \log\cos x$ is strictly concave on $(-\pi/2, \pi/2)$, since $f''(x) = -\sec^2 x < 0$. Applying Jensen's inequality to $-f$ and $\theta_1, ..., \theta_n$ we obtain

$$\log\left(\prod_{j=1}^{n} (\cos\theta_j)^{p_j}\right) = \sum_{j=1}^{n} p_j f(\theta_j) \leq f\left(\sum_{j=1}^{n} p_j\theta_j\right) = \log\cos\left(\sum_{j=1}^{n} p_j\theta_j\right),$$

or equivalently,

$$\prod_{j=1}^{n} (\cos\theta_j)^{p_j} \leq \cos\left(\sum_{j=1}^{n} p_j\theta_j\right).$$

To prove

$$\left|\prod_{j=1}^{n} (1 + z_j)^{p_j} - 1\right| < 1,$$

it suffices to show that

$$\prod_{j=1}^{n} r_j^{p_j} < 2\cos\left(\sum_{j=1}^{n} p_j\theta_j\right).$$

This is clearly the case, since

$$\prod_{j=1}^{n} r_j^{p_j} < \prod_{j=1}^{n} (2\cos\theta_j)^{p_j} = 2\prod_{j=1}^{n} (\cos\theta_j)^{p_j} \le 2\cos\left(\sum_{j=1}^{n} p_j\theta_j\right).$$

This completes the proof. □

Corollary 7 (Hölder's inequality)**.** *Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be positive real numbers. Suppose that $p, q > 1$ are positive real numbers such that $1/p + 1/q = 1$. Then*

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

with equality if and only if $a_1^p/b_1^q = ... = a_n^p/b_n^q$.

Proof. Let $p_1, ..., p_n$ be positive real numbers with $\sum_{i=1}^{n} p_i = 1$. Applying Jensen's inequality to $f(x) := x^q$ yields

$$\left(\sum_{i=1}^{n} p_i x_i\right)^q \le \sum_{i=1}^{n} p_i x_i^q$$

for any $x_1, ..., x_n > 0$ with equality if and only if $x_1 = ... = x_n$. Taking $x_i = a_i^{-p/q} b_i$ and

$$p_i = \frac{a_i^p}{\sum_{j=1}^{n} a_j^p}$$

for all $1 \le i \le n$ and observing that $p/q = p - 1$, we obtain

$$\left(\sum_{i=1}^{n} a_i b_i\right)^q \left(\sum_{i=1}^{n} a_i^p\right)^{-q} \le \left(\sum_{i=1}^{n} b_i^q\right) \left(\sum_{i=1}^{n} a_i^p\right)^{-1},$$

which can be rewritten as

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$

Equality occurs if and only if $a_1^{-p/q} b_1 = ... = a_n^{-p/q} b_n$, or equivalently, if and only if $a_1^p/b_1^q = ... = a_n^p/b_n^q$. □

Remark 3. The standard proof of Hölder's inequality uses Young's inequality which is an equivalent formulation of the arithmetic mean-geometric mean inequality for $n = 2$, whereas our proof uses the convexity of the power function x^q.

Corollary 8. *Let $I \subseteq \mathbb{R}$ be an interval which is also a semigroup, and let $f : I \to \mathbb{R}$ be a convex function. Suppose that $X = (x_{ij}) \in M_{m\times n}(I)$ is an $m \times n$ matrix with the property that $x_{i1} \ge ... \ge x_{in}$ for every $i = 1, ..., m$. If $\sigma_1, ..., \sigma_m$ are permutations of $\{1, ..., n\}$, then*

$$\sum_{j=1}^{n} f\left(\sum_{i=1}^{m} x_{ij}\right) \ge \sum_{j=1}^{n} f\left(\sum_{i=1}^{m} x_{i\sigma_i(j)}\right).$$

Proof. For each $1 \leq j \leq n$, let

$$a_j := \sum_{i=1}^{m} x_{ij},$$
$$b_j := \sum_{i=1}^{m} x_{i\sigma_i(j)}.$$

Then $a_1 \geq ... \geq a_n$. Let τ be a permutation of $\{1, ..., n\}$ such that $b_{\tau(1)} \geq ... \geq b_{\tau(n)}$. Then

$$\sum_{j=1}^{k} a_j = \sum_{i=1}^{m}\sum_{j=1}^{k} x_{ij} \geq \sum_{i=1}^{m}\sum_{j=1}^{k} x_{i\sigma_i(\tau(j))} = \sum_{j=1}^{k} b_{\tau(j)}$$

for all $1 \leq k \leq n-1$ and

$$\sum_{j=1}^{n} a_j = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i\sigma_i(\tau(j))} = \sum_{j=1}^{n} b_{\tau(j)}.$$

Our corollary follows now from Karamata's inequality applied to $a_1, ..., a_n, b_1, ..., b_n$. □

Remark 4. Two interesting special cases of Corollary 8 are

$$\sum_{j=1}^{n}\prod_{i=1}^{m} x_{ij} \geq \sum_{j=1}^{n}\prod_{i=1}^{m} x_{i\sigma_i(j)},$$
$$\prod_{j=1}^{n}\sum_{i=1}^{m} x_{ij} \leq \prod_{j=1}^{n}\sum_{i=1}^{m} x_{i\sigma_i(j)},$$

where $x_{ij} > 0$ for all pairs (i, j).

Corollary 9. *Let $P(x) \in \mathbb{R}[x]$ be a non-constant polynomial with only real zeros such that $P(x) > 0$ for all $x \in [a, b]$. Let $p_1, ..., p_n$ be positive real numbers. Suppose that $x_1, ..., x_n \in [a, b]$ and $y_1, ..., y_n \in [a, b]$ are two sequences of real numbers arranged in descending order satisfying (9) and (10). Then*

$$\prod_{i=1}^{n} (P(x_i))^{p_i} \leq \prod_{i=1}^{n} (P(y_i))^{p_i}$$

with equality if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. Let us write

$$P(x) = A \cdot \prod_{i=1}^{m} (x - c_i),$$

where $m \geq 1$ and $c_1, ..., c_m, A \in \mathbb{R}$ with $A \neq 0$. The function $f(x) := \log P(x)$ is well-defined and strictly concave on $[a, b]$, since

$$f''(x) = -\sum_{i=1}^{m} \frac{1}{(x - c_i)^2} < 0.$$

Our corollary follows now from Karamata's inequality applied to $-f$. □

Corollary 10. *Let $r \geq s \geq 0$ be non-negative integers. Let $I, J \subseteq \mathbb{R}$ be two disjoint intervals, and let $a_1, ..., a_r \in J$ and $b_1, ..., b_s \in J$ be two sequences of real numbers arranged in descending order satisfying*

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i$$

for all $1 \leq k < s$ and

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} b_i.$$

Let $R(x)$ be a non-constant rational function with real coefficients having zeros $a_1, ..., a_r$ and poles $b_1, ..., b_s$ such that $R(x) > 0$ for all $x \in I$. Let $p_1, ..., p_n$ be positive real numbers, and suppose that $x_1, ..., x_n \in I$ and $y_1, ..., y_n \in I$ are two sequences of real numbers arranged in descending order satisfying (9) and (10). Then

$$\prod_{i=1}^{n} (R(x_i))^{p_i} \leq \prod_{i=1}^{n} (R(y_i))^{p_i}$$

with equality if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. We may write

$$R(x) = A \cdot \frac{(x - a_1) \cdots (x - a_r)}{(x - b_1) \cdots (x - b_s)},$$

where $A \in \mathbb{R} \setminus \{0\}$. Consider the function $f(x) := \log R(x)$ for $x \in I$. Note that

$$f''(x) = -\sum_{i=1}^{r} \frac{1}{(x - a_i)^2} + \sum_{i=1}^{s} \frac{1}{(x - b_i)^2}.$$

For each $x \in I$, let $g_x(y) := 1/(y - x)^2$ for $y \in J$. Then g_x is strictly convex on J. We have by Karamata's inequality that for every $x \in I$,

$$f''(x) = -\sum_{i=1}^{r} g_x(a_i) + \sum_{i=1}^{s} g_x(b_i) \leq -\sum_{i=1}^{s} g_x(a_i) + \sum_{i=1}^{s} g_x(b_i) \leq 0.$$

Since $R(x)$ is non-constant, it follows that either $r > s$ or $r = s$ with $a_i \neq b_i$ for some $1 \leq i \leq r$, which implies that the above inequality concerning $f''(x)$ is strict for every $x \in I$. Thus $f(x)$ is strictly concave on I. Our corollary follows now by applying Karamata's inequality to $-f$. □

Remark 5. It is evident that Corollary 9 follows from Corollary 10 by taking $r > s = 0$.

Corollary 11. *Let $P(x) \in \mathbb{R}[x]$ be a non-constant polynomial with non-negative real coefficients. Let $p_1, ..., p_n$ be positive real numbers. Suppose that $x_1, ..., x_n$ and $y_1, ..., y_n$ are two sequences of positive real numbers arranged in descending order satisfying*

$$\prod_{i=1}^{k} x_i^{p_i} \geq \prod_{i=1}^{k} y_i^{p_i} \tag{15}$$

for all $1 \leq k \leq n$. Then

$$\prod_{i=1}^{n} (P(x_i))^{p_i} \geq \prod_{i=1}^{n} (P(y_i))^{p_i}. \tag{16}$$

Moreover, if $P(x)$ is not a constant multiple of x^m for some $m \geq 1$, then equality in (16) holds if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. Suppose that

$$P(x) = \sum_{k=0}^{m} a_k x^k,$$

where $a_0, ..., a_{m-1} \geq 0$ and $a_m > 0$. We claim that for any $x > 0$ we have

$$(P''(x)y + P'(x))P(x) \geq P'(x)^2 x$$

which becomes strict when $a_k > 0$ for some $0 \leq k \leq m-1$. To prove this, note that

$$P''(x)x + P'(x) = \sum_{k=2}^{m} k(k-1)a_k x^{k-1} + \sum_{k=1}^{m} k a_k x^{k-1} = \sum_{k=1}^{m} k^2 a_k x^{k-1}.$$

It follows by Cauchy-Schwarz inequality that

$$\begin{aligned}(P''(x)x + P'(x))P(x) &= \left(\sum_{k=1}^{m} k^2 a_k x^{k-1}\right)\left(\sum_{k=0}^{m} a_k x^k\right) \\ &\geq \left(\sum_{k=1}^{m} k^2 a_k x^{k-1}\right)\left(\sum_{k=1}^{m} a_k x^{k-1}\right) x \\ &\geq \left(\sum_{k=1}^{m} k a_k x^{k-1}\right)^2 x \\ &= P'(x)^2 x.\end{aligned}$$

If $a_0 > 0$, then

$$\sum_{k=0}^{m} a_k x^k > \left(\sum_{k=1}^{m} a_k x^{k-1}\right) y;$$

if $a_k > 0$ for some $1 \leq k \leq m-1$, then

$$\left(\sum_{k=1}^{m} k^2 a_k x^{k-1}\right)\left(\sum_{k=1}^{m} a_k x^{k-1}\right) > \left(\sum_{k=1}^{m} k a_k x^{k-1}\right)^2.$$

In both cases, we must have

$$(P''(x)x + P'(x))P(x) > P'(x)^2 x.$$

This proves our claim.

Consider now the function $f(x) := \log P(e^x)$, where $x \in \mathbb{R}$. Since $P(x)$ is non-constant with non-negative real coefficients, we see that $f'(x) = P'(e^x)e^x/P(e^x) > 0$ for all $x \in \mathbb{R}$. Simple calculations show that

$$f''(x) = \frac{[(P''(e^x)e^x + P'(e^x))P(e^x) - P'(e^x)^2 e^x]e^x}{P(e^x)^2} \geq 0$$

for all $x \in \mathbb{R}$. Thus f is convex on $\mathbb{R}$. Furthermore, if $P(x)$ is not a constant multiple of x^m, or equivalently, if $a_k > 0$ for some $0 \leq k \leq m-1$, then $f''(x) > 0$ for all $x \in \mathbb{R}$, which implies that f is strictly convex on $\mathbb{R}$. Our corollary follows immediately from Proposition 4 applied to f and $\log x_1, ..., \log x_n, \log y_1, ..., \log y_n$. □

Corollary 12. *Let $p_1, ..., p_n$ be positive real numbers and suppose that $x_1, ..., x_n$ and $y_1, ..., y_n$ are two sequences of positive real numbers arranged in descending order satisfying (15) for all $1 \leq k \leq n$. Then*

$$\prod_{i=1}^{n} (\log(1+x_i))^{p_i} \leq \prod_{i=1}^{n} (\log(1+y_i))^{p_i}$$

with equality if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. Let $f(x) := \log\log(1+e^x)$, where $x \in \mathbb{R}$. Then f is strictly increasing on $\mathbb{R}$ with

$$f''(x) = \frac{e^x[\log(1+e^x) - e^x]}{[(1+e^x)\log(1+e^x)]^2} < 0,$$

where we have used the fact that $\log(1+y) \leq y$ for any $y > -1$ with equality precisely when $y = 0$. This shows that f is strictly concave on $\mathbb{R}$. We finish the proof of our corollary by applying Proposition 4 to $-f$ and $\log x_1, ..., \log x_n, \log y_1, ..., \log y_n$. ☐

Corollary 13. *Let $p_1, ..., p_n$ be positive real numbers and suppose that $x_1, ..., x_n$ and $y_1, ..., y_n$ are two sequences of positive real numbers arranged in descending order satisfying (9) and (10). Then*

$$\prod_{i=1}^{n} (\Gamma(x_i))^{p_i} \geq \prod_{i=1}^{n} (\Gamma(y_i))^{p_i},$$
$$\sum_{i=1}^{n} p_i\Gamma(x_i) \geq \sum_{i=1}^{n} p_i\Gamma(y_i),$$

with equality if and only if $x_i = y_i$ for all $1 \leq i \leq n$, where Γ is the Gamma function *defined for every $x > 0$ by*

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Proof. This follows directly from Karamata's inequality and the well-known fact that both $\Gamma(x)$ and $\log\Gamma(x)$ are strictly convex on $(0, +\infty)$. ☐

Remark 6. It is well known that for any $x \in (0,1)$ we have

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{17}$$

This implies that the function $f(x) := \Gamma(x)\Gamma(1-x)$ is strictly decreasing on $(0, 1/2]$ and strictly increasing on $[1/2, 1)$. Another way to see this is to use Corollary 13. Indeed, note that $f(x)$ is symmetric about the line $x = 1/2$. It thus suffices to consider the interval $(1/2, 1]$. If $1/2 \leq y < x \leq 1$, then Corollary 13 applied with $(x_1, x_2) = (x, 1-x)$ and $(y_1, y_2) = (y, 1-y)$ gives $f(x) > f(y)$.

Corollary 13 can also be used to produce numerical bounds for special values of Γ. For instance, we obtain by taking $(x_1, x_2) = (n+1, n+1/3)$, $(y_1, y_2) = (n+2/3, n+1/2)$ and $(p_1, p_2) = (1, 2)$ with $n \geq 1$ that

$$n! \cdot \Gamma\left(n + \frac{1}{3}\right)^2 > \Gamma\left(n + \frac{2}{3}\right)\Gamma\left(n + \frac{1}{2}\right)^2.$$

Note that

$$\Gamma(n+\alpha)=\Gamma(\alpha)\prod_{k=1}^{n}(k-1+\alpha)$$

for any $\alpha\in(0,1)$. Thus we have

$$\Gamma\left(\frac{1}{3}\right)^2>\Gamma\left(\frac{2}{3}\right)\pi\cdot\prod_{k=1}^{n}\left(1-\frac{1}{3k}\right)\left(1+\frac{1}{6k-4}\right)^2.$$

Since

$$\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{\sin(2\pi/3)}\Gamma\left(\frac{1}{3}\right)^{-1}=\frac{2\pi}{\sqrt{3}}\Gamma\left(\frac{1}{3}\right)^{-1}$$

by (17), we obtain by letting $n\to\infty$ that

$$\Gamma\left(\frac{1}{3}\right)\geq\sqrt[3]{\frac{2\pi^2A}{\sqrt{3}}},$$

where

$$A:=\prod_{k=1}^{\infty}\left(1-\frac{1}{3k}\right)\left(1+\frac{1}{6k-4}\right)^2.$$

Similarly, we get by taking $(x_1,x_2)=(n+3/2,n+1)$, $(y_1,y_2)=(n+4/3,n+4/3)$ and $(p_1,p_2)=(2,1)$ that

$$n!\cdot\Gamma\left(n+\frac{3}{2}\right)^2>\Gamma\left(n+\frac{4}{3}\right)^3,$$

which is equivalent to

$$\Gamma\left(\frac{1}{3}\right)^3<3\pi\left(1+\frac{1}{6n+2}\right)^2\prod_{k=1}^{n}\left(1-\frac{1}{3k+1}\right)\left(1+\frac{1}{6k-4}\right)^2.$$

Letting $n\to\infty$ we deduce that

$$\Gamma\left(\frac{1}{3}\right)\leq\sqrt[3]{3\pi B},$$

where

$$B:=\prod_{k=1}^{\infty}\left(1-\frac{1}{3k+1}\right)\left(1+\frac{1}{6k-4}\right)^2.$$

Numerical data shows that

$$\sqrt[3]{\frac{2\pi^2A}{\sqrt{3}}}=2.678938534707747633655692940974677644128689377957301100950428327566...,$$

$$\sqrt[3]{3\pi B}=2.678938534707747633655692940974677644128689377957301100950428327584....$$

We see that the bounds we have obtained are fairly good, especially considering that in deriving these bounds we made no use of the deep properties of Γ other than Corollary 13 and the reflection formula (17).

Corollary 14. *Let $p_1, ..., p_n$ positive real numbers. Suppose that $a_1, ..., a_n$ and $b_1, ..., b_n$ are two sequences of positive real numbers arranged in descending order satisfying (9) and (10). Let $c_1, ..., c_n$ and $d_1, ..., d_n$ be another two sequences of positive real numbers satisfying the same conditions. Then*

$$\sum_{i=1}^{n} p_i B(a_i, c_i) \geq \sum_{i=1}^{n} p_i B(b_i, d_i)$$

with equality if and only if $(a_i, c_i) = (b_i, d_i)$ for all $1 \leq i \leq n$, where $B(x, y)$ is the Beta function *defined for every pair $(x, y) \in (0, +\infty)^2$ by*

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

Proof. Let $f_i(x) := B(x, c_i)$ for every $1 \leq i \leq n$. Then

$$f_i'(x) = \int_0^1 t^{x-1}(1-t)^{c_i-1} \log t\, dt < 0,$$

$$f_i''(x) = \int_0^1 t^{x-1}(1-t)^{c_i-1} (\log t)^2\, dt > 0.$$

It follows that each $f_i(x)$ is strictly convex on $(0, +\infty)$ and $f_1'(x) \geq ... \geq f_n'(x)$ holds for all $x > 0$. Thus Proposition 5 applies; we obtain

$$\sum_{i=1}^{n} p_i B(a_i, c_i) \geq \sum_{i=1}^{n} p_i B(b_i, c_i)$$

with equality if and only if $a_i = b_i$ for all $1 \leq i \leq n$. Similarly, we have

$$\sum_{i=1}^{n} p_i B(c_i, b_i) \geq \sum_{i=1}^{n} p_i B(d_i, b_i)$$

with equality if and only if $c_i = d_i$ for all $1 \leq i \leq n$. Combining these two inequalities and using the identity $B(x, y) = B(y, x)$ we obtain

$$\sum_{i=1}^{n} p_i B(a_i, c_i) \geq \sum_{i=1}^{n} p_i B(b_i, d_i)$$

with equality if and only if $(a_i, c_i) = (b_i, d_i)$ for all $1 \leq i \leq n$. □

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